# NUMERICAL ALGORITHMS FOR INVERSE AND ILL-POSED PROBLEMS

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## Summary

This chapter is devoted to inverse and ill-posed problems and numerical algorithms for their solution. Inverse and ill-posed problems arise in science, engineering, medicine, ecology, etc. Development of the theory of inverse and ill-posed problems is strongly connected with application of mathematical methods for processing and interpreting the results of observations. Mathematical methods and mathematical models are used for research on various processes. Parameters of a mathematical model are characteristics of a process that is being investigated. Inverse problems arise when some parameters of a mathematical model are unknown. We have to determine unknown characteristics of a process by its indirect manifestation that is the result of observation. In solving an inverse problem, we try to invert cause and effect connection. As a consequence, inverse problems are usually ill-posed problems. Numerical algorithms for solving ill-

posed problems must take into account their peculiarities. Major development of the theory of inverse problems and numerical algorithms for solution of ill-posed problems took place in the second half of the 20th century.

## 1. Introduction

Research on various processes can be conducted by using mathematical models. In the most cases the description of the mathematical model is given in terms of a set of equations containing certain parameters. Parameters are characteristics of the process, which is investigated. These characteristics determine the process. Therefore, they are the cause of the process. If we know the characteristics, then we can solve the mathematical model and obtain a solution. This solution gives a description of the process. Of course, the solution is determined by characteristics. Therefore, the solution is the effect.  If we study the mathematical model with given characteristics, then we solve the following problem. Determine the effect if the cause is given. This problem is called a direct problem.

Let us consider the following example of a direct problem. A particle with a given mass moves in space. The motion is governed by force, which depends on time. The force is the cause of the process of motion. If we know the force, the initial position and the initial speed of the particle, then we can calculate the particle position at any time using Newton law. The position of the particle is the effect of the action of the force. This direct problem is one of the problems of classical mechanics. Direct problems were typical for the scientific research for a long time.

There exists another important class of the problems for mathematical models. These problems arise in the cases when some characteristics of the process are unknown. The general feature of this class of problems is that we have to determine the unknown characteristics by their indirect manifestation that can be measured experimentally. Therefore, we know from experiment only the effect and we need to determine the cause. These problems are called inverse problems. For example, we may consider the following inverse problem for the process of motion of a particle. The force acting on the particle is unknown. In experiment we measure the position of particle at any time. We have to determine the force (cause) if the position of particle (effect) is given.

Various inverse problems arise in science, engineering, medicine, ecology, etc. The theory of inverse problems has great importance in processing and interpretation of observations. By applying mathematical methods to solve inverse problems we can obtain new information on the investigated process.

When we solve the inverse problem we invert the connection between cause and effect. As a consequence, setting of inverse problems is not so good from the classical point of view. Usually, the solution of an inverse problem is unstable with respect to initial data. The important peculiarity of the inverse problems arising in experiments is that the initial information is given only approximately. Therefore, we must create special numerical algorithms that are stable with respect to initial data.

## 2. Inverse Problems

Inverse problems can be considered for various mathematical models. Setting of an inverse problem depends on the mathematical model, unknown characteristics of the model and initial information, which is given.

### 2.1 Reconstruction of Input Signal

The action of many physical devices can be described by the following mathematical model

$$\int_a^b K(x,s)z(s)ds = u(x), \quad c \le x \le d \,. \tag{1}$$

Here $z(s)$ is input signal, $u(x)$ is output signal and $K(x,s)$ is often called apparatus function. If the input signal (cause) and apparatus function are given, then one can calculate integral

$$\int_a^b K(x,s)z(s)ds,$$

and obtain output signal (effect). The following inverse problem arises very often in scientific research. Determine input signal (cause) $z(s)$ if output signal $u(x)$ (effect) and apparatus function $K(x,t)$ are given. This inverse problem is the problem of solving integral Eq. (1).

### 2.2 Inverse Problems for Ordinary Differential Equations

Ordinary differential equations or systems of ordinary differential equations are used for the description of various natural and social processes. If all the coefficients of an equation are given, then it is possible to find a solution of this equation satisfying some initial or boundary conditions. Another problem arises very often in science and engineering. Some coefficients of the equation are unknown. One has to determine these coefficients if a few solutions of the equation are given. This problem is an inverse problem for ordinary differential equations.

Let us consider examples of inverse problems for linear ordinary differential equations of the form

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + ... + a_1(x)y'(x) + a_0(x)y(x) = f(x), c \le x \le d \,. \tag{2}$$

One of the inverse problems for Eq. (2) is the problem of determination of unknown function $f(x)$ if all coefficients $a_i(x)$ and a solution of Eq. (2) $y_0(x)$ are given. Another inverse problem is the problem of determination of a coefficient $a_k(x)$ if all other coefficients $a_i(x), i = 0,1,...k-1,k+1,...n-1$ , $f(x)$ and a solution of equation $y_0(x)$ are given. It is possible to consider a more complicated inverse problem. To determine $m$ unknown coefficients $a_i(x)$, if other $n-m$ coefficients $a_j(x)$, function $f(x)$ and $m$ solutions of Eq. (2) are given. The following important question arises in the study of this inverse problem. Can one determine uniquely $m$ unknown coefficients if $m$ solutions of Eq. (2) are given? Positive answer for this question depends on solutions, which are given. Let us consider the following example. In the ordinary differential equation

$$y^{(4)}(x) + a_2(x)y''(x) + a_0(x)y(x) = 0 \qquad (3)$$

coefficients $a_3(x) = a_1(x) = 0,\ \ f(x) = 0$ and coefficients $a_2(x), a_0(x)$ are unknown. Let $y_1(x) = \sin x,\ y_2(x) = \cos x$ be two solutions of Eq. (3). Substitution $y_1(x), y_2(x)$ in equation (3) gives only one equation for unknown coefficients $a_2(x) - a_0(x) = 1$. So the solution of this inverse problem is not unique. If another couple of solutions $y_1(x) = \exp(x),\ y_2(x) = \exp(2x)$ of Eq. (3) is given, then substitution gives two equations for unknown coefficients $a_2(x) + a_0(x) = -1$ and $4a_2(x) + a_0(x) = -16$. This system of equations has a unique solution $a_2(x) = -5,\ \ a_0(x) = 4$. These examples show the importance of analysis of information, which we use to solve the inverse problem.

Inverse problems for systems of differential equations can be considered also. Let

$$\frac{d\vec{x}}{dt} = A\vec{x}(t), \quad t_0 \le t \le T \qquad (4)$$

be a system of linear differential equations of $n$-th order, where $A$ is a matrix with unknown constant elements $a_{ij}, 1 \le i, j \le n.$ One might study the following inverse problem. Determine elements $a_{ij}, 1 \le i, j \le n$ if a vector solution $\vec{x}_0(t)$ of the system (4) is given for $t_0 \le t \le T$. Inverse problems like this arise in many scientific fields, for example, in studying chemical kinetic processes. Other inverse problems are considered for system of ordinary differential equations (4) with coefficients, which are functions of $t$: $a_{ij} = a_{ij}(t)$. Let some coefficients $a_{ij}(t)$ be given and the others be unknown. We have to determine unknown coefficients if a few vector solutions $\vec{x}_k(t), k = 1...m$ are given.

There exists another important class of inverse problems for ordinary differential equations with parameters. These are inverse spectral problems. Let us consider inverse the Sturm- Liouville problem, which is a significant example of inverse spectral problems. The classical Sturm-Liouville problem consists in determining the values $\lambda$ such that boundary value problem

$$y''(x) + (\lambda - q(x))y(x) = 0, \quad 0 \le x \le l, \qquad (5)$$

$$y(0)-hy'(0)=0, \tag{6}$$

$$y(l)+Hy'(l)=0, \tag{7}$$

has a nontrivial solution $y(x)$. These values $\lambda$ are called eigenvalues. For a given function $q(x)$ there exists an infinite set of eigenvalues $\{\lambda_n\}$ $n=1,2,\ldots$. These eigenvalues are real and tend to $+\infty$ as $n \to +\infty$. If one replaces the boundary condition in (7) by

$$y(l)+H_1 y'(l)=0, \tag{8}$$

then problem consisting of (5),(6),(8) has another set of eigenvalues $\{\mu_n\}$ $n=1,2,\ldots$. Let us formulate the inverse Sturm-Liouville problem. Determine the unknown function $q(x)$, if two sets of eigenvalues $\{\lambda_n\},\{\mu_n\}$ $n=1,2,\ldots$ are given. Solution of this inverse problem is unique. If only one set of eigenvalues $\{\lambda_n\}$, $n=1,2,\ldots$ is given, then solution of inverse problem is nonunique. There are other important inverse spectral problems: reconstruction of $q(x)$ by spectral function, inverse problem of quantum scattering theory.

### 2.3 Inverse Problems for Partial Differential Equations

Many subject areas in natural science, engineering, medicine and ecology are dominated by the study of mathematical models based on partial differential equations. Let us consider as an example the following problem for one dimensional parabolic equation

$$c(x)u_t(x,t)=(k(x)u_x(x,t))_x-q(x)u(x,t)+f(x,t),\ 0<x<l, 0<t\le T\ , \tag{9}$$

$$u(0,t)=p(t), \quad 0\le t\le T, \tag{10}$$

$$-k(l)u_x(l,t)=h(t), \quad 0\le t\le T, \tag{11}$$

$$u(x,0)=\varphi(x), \quad 0\le x\le l, \tag{12}$$

which is the mathematical model for different physical processes. The coefficients in the equation are certain characteristics of the process investigated. For example, if problem (9)-(12) describes the process of heat transfer in a rod, then coefficients $c(x)$ and $k(x)$ are the heat capacity and heat conductivity coefficients respectively. They characterize the material of the rod. All the other functions in Eq. (9), boundary conditions (10),(11) and initial condition (12) have a thermophysical sense. In the framework of the given mathematical model of heat transfer, the temperature in the rod at a point $x$ at the moment $t$, the function $u(x,t)$, which solves problem (9)-(12) is determined by the characteristics of this model: $c,k,q,f,p,h,\varphi$. In the case when these functions are given, one can solve problem (9)-(12), namely, to find the function $u(x,t)$, that is to describe the process of heat transfer in the rod. However, in many physical processes, some characteristics are unknown, but from experiment one can obtain certain additional information on temperature. For example, all the functions, which determine mathematical model (9)-(12) are known except the heat conductivity coefficient $k(x)$. From experiment one can determine the function

$$b(t) = u(x_0, t), \quad 0 \le t \le T. \tag{13}$$

This function is the temperature at a certain interior point of the rod, as function of time. Thus, there arises the following inverse problem: to determine the heat conductivity coefficient $k(x)$, if the function $b(t)$ is given. It is possible to set up other inverse problems for mathematical model (9)-(12): to determine the heat capacity coefficient $c(x)$, if all other functions in (9)-(12) and $b(t)$ are given; to determine heat source $f(x,t)$, if all other functions in (9)-(12) and $b(t)$ are given; to determine boundary temperature $p(t)$, if all other functions in (9)-(12) and $b(t)$ are given and so on. The variety of possible inverse problems is determined not only by unknown functions in mathematical model, but also by different ways of giving the additional information, that is by the character of experiment. For example, instead of (13) the heat flux $-k(0)u_x(0,t) = g(t), 0 \le t \le T$ or temperature $u(x,T) = d(x), 0 \le x \le l$ can be given.

Various inverse problems for other partial differential equations can be considered in the same way as for Eq. (9). Of course, setting up of inverse problem depends on the type of equation, unknown function and additional information, which is given.

### 2.4 Computerized Tomography Problem

Computerized tomography arose as medical tomography was developed very intensively. Now tomographic methods have been used not only in medicine; but they have also been applied to solve a range of other scientific and technical problems. The essence of the tomographic technique for X-ray diagnostic is as follows.

A narrow linear X-ray beam is focused onto a planar section of the body. The change in intensity of radiation as the beam penetrates the body is recorded by a detector. Similar measurements are made for all different directions of X-ray beam. Using the values of change of intensity, one can evaluate the following integral

$$\int_L f(x,y)\,dl, \tag{14}$$

where the function $f(x,y)$ represents the coefficient of absorption of X-ray and $L$ is the straight line corresponding to direction of the beam. As integral (14) is evaluated for different directions (different straight lines) the problem can be reduced to one of determining the function $f(x,y)$ from its integral (14) taken over each straight line in the plane. A straight line in the plane may be given as follows $x\cos\varphi + y\sin\varphi = p$. Thus, the set of lines in the plane is the two parameter set $L(p,\varphi)$, where $-\infty < p < +\infty, 0 \le \varphi \le 2\pi$. Consequently integral (14) is function of parameters

$p$ and $\varphi$

$$\int\limits_{L(p,\varphi)} f(x,y)dl = u(p,\varphi), \tag{15}$$

which is measured in the experiment. Hence the computerized tomography problem is the following inverse problem. Determine the unknown function $f(x,y)$, if integrals (15) are given for all $p,\varphi$ .

Let us consider the special set of unknown functions: $f(x,y) = f(\sqrt{x^2+y^2}),\ x^2+y^2 \le a^2$ and $f(x,y)=0,\ x^2+y^2 > a^2$. In this case the function $u(p,\varphi)$ does not depend on $\varphi$ and

$$\int\limits_p^a \frac{2f(s)sds}{\sqrt{s^2-p^2}} = u(p), \quad 0 \le p \le a. \tag{16}$$

Thus the computerized tomography problem in this case is reduced to solving the integral Eq. (16) with given function $u(p)$ and unknown $f(s)$.

### 2.5 Abstract Setting of Inverse Problems

All the inverse problems considered above may be described by a general abstract formulation. Denote by $z$ the unknown characteristic of the mathematical model of process and by operator $A$, an operator, which maps $z$ (the cause) into $u$ (the effect), the result of experiment. So, the inverse problem is to solve the operator equation

$$Az = u, \tag{17}$$

where operator $A$ and element $u$ are given and $z$ is unknown. For the inverse problem of restoration of input signal $z$ is the input signal, $u$ is the output signal and $A$ is the integral operator, which is defined by the apparatus function $K(x,s)$. For the inverse Sturm-Liouville problem $z = q(x)$, $u$ is two sets of eigenvalues $\{\lambda_n\},\{\mu_n\},\ n = 1,2,\ldots$. The operator $A$ is defined by (5), (6), (7) and (5), (6), (8).

For one of the inverse problems for the mathematical model of the heat transfer process (9)-(12) $z = k(x)$ is the heat conductivity coefficient; $A$ is the operator determined by (9)-(13). This operator, for all fixed remaining parameters of the model $\{c(x), q(x), F(x,t), p(t), h(t), \varphi(x)\}$ puts into correspondence to all different $k(x)$ the different solution of (9)-(12) at the point $x_0$, that is, $u(x_0,t)$.

Element $u$ for this inverse problem is $b(t)$. For computerized tomography problem $z = f(x,y), u = u(p,\varphi)$ and the operator $A$ is determined by the expression in the left-hand side of formula (15).

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## Biographical Sketch

**A.M. Denisov** was born in Moscow, Russia. He completed his studies with the Ph.D in Mathematics at the Lomonosov Moscow State University (LMSU) for the thesis on solving integral equations of the first kind in 1972. In 1987 he obtained the Russian degree of Doctor in Physics and Mathematics at LMSU for the thesis 'Inverse problems of heat conduction, absorption, scattering and methods for solving them'. From 1972 to 1994 Dr. A. Denisov was assistant professor, associate professor and professor of the Faculty of Computational Mathematics and Cybernetics of the LMSU. Since 1994 he is Head of Department of Mathematical Physics of the Faculty of Computational Mathematics and Cybernetics of the LMSU. Prof. A. Denisov's special fields of research activity include theory of inverse problems of mathematical physics, theory of ill-posed problems and integral equations. He wrote about 100 scientific papers including 3 books. He is a member of Editorial Boards of 3 Scientific Journals.